Linear Algebra I

13/04/2017, Thursday, 18:30 – 21:30

You are **NOT** allowed to use any type of calculators.

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1 $(4+2+7+2=15$ pts$)$ **Linear systems of equations**

In this problem, we want to find A, B, and C that satisfy the equation

$$\frac{2x^2-4}{(x-1)^3} = \frac{A}{x-1} + \frac{B(x-2)}{(x-1)^2} + \frac{C(x-3)}{(x-1)^3}$$

for all $x \in \mathbb{R}$.

a. Find the linear equations that A, B, and C should satisfy.

b. Write down the augmented matrix for the linear equations.

c. Put the augmented matrix into row reduced echelon form.

d. Find the general solution.

REQUIRED KNOWLEDGE: **Gauss-elimination, row operations, (reduced) row echelon form, general solution.**

SOLUTION:

1a: We have

$$\begin{aligned}
2x^2-4 &= A(x-1)^2 + B(x-2)(x-1) + C(x-3)\\
2x^2-4 &= A(x^2-2x+1) + B(x^2-3x+2) + C(x-3)\\
2x^2-4 &= (A+B)x^2 + (-2A-3B+C)x + (A+2B-3C).
\end{aligned}$$

Therefore, we obtain the linear equation system:

$$\begin{aligned}
A+B &= 2\\
-2A-3B+C &= 0\\
A+2B-3C &= -4.
\end{aligned}$$

1b: The augmented matrix for the equation above is given by

$$\begin{bmatrix} 1 & 1 & 0 & \vdots & 2\\ -2 & -3 & 1 & \vdots & 0\\ 1 & 2 & -3 & \vdots & -4 \end{bmatrix}.$$

1c: By applying row operations, we obtain:

$$\begin{bmatrix} 1 & 1 & 0 & \vdots & 2\\ -2 & -3 & 1 & \vdots & 0\\ 1 & 2 & -3 & \vdots & -4 \end{bmatrix} \xrightarrow[\textbf{3rd = 3rd} - \textbf{1st}]{\textbf{2nd = 2nd} + 2\times\textbf{1st}} \begin{bmatrix} 1 & 1 & 0 & \vdots & 2\\ 0 & -1 & 1 & \vdots & 4\\ 0 & 1 & -3 & \vdots & -6 \end{bmatrix} \xrightarrow{\textbf{2nd} = -\textbf{2nd}} \begin{bmatrix} 1 & 1 & 0 & \vdots & 2\\ 0 & 1 & -1 & \vdots & -4\\ 0 & 1 & -3 & \vdots & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & \vdots & 2 \\ 0 & 1 & -1 & \vdots & -4 \\ 0 & 1 & -3 & \vdots & -6 \end{bmatrix} \xrightarrow{\textbf{3rd} = \textbf{3rd} - \textbf{2nd}} \begin{bmatrix} 1 & 1 & 0 & \vdots & 2 \\ 0 & 1 & -1 & \vdots & -4 \\ 0 & 0 & -2 & \vdots & -2 \end{bmatrix} \xrightarrow{\textbf{3rd} = -\frac{1}{2}\textbf{3rd}} \begin{bmatrix} 1 & 1 & 0 & \vdots & 2 \\ 0 & 1 & -1 & \vdots & -4 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}.$$

So far, we obtained a matrix in row echelon form. Continuing row operations, we get

$$\begin{bmatrix} 1 & 1 & 0 & \vdots & 2 \\ 0 & 1 & -1 & \vdots & -4 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix} \xrightarrow{\textbf{2nd} = \textbf{2nd} + \textbf{3rd}} \begin{bmatrix} 1 & 1 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix} \xrightarrow{\textbf{1st} = \textbf{1st} - \textbf{2nd}} \begin{bmatrix} 1 & 0 & 0 & \vdots & 5 \\ 0 & 1 & 0 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}.$$

1d: From the reduced row echelon form, we see that the general solution is $A = 5$, $B = -3$, and $C = 1$.

Find all values of a and b such that the matrix

$$\begin{bmatrix} 1 & a & b \\ 1 & a^2 & b^2 \\ 1 & a^3 & b^3 \end{bmatrix}$$

is nonsingular.

REQUIRED KNOWLEDGE: **Determinant and nonsingularity**

SOLUTION:

By applying row/column operations, we obtain

$$\det\left(\begin{bmatrix} 1 & a & b \\ 1 & a^2 & b^2 \\ 1 & a^3 & b^3 \end{bmatrix}\right) = ab\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & a^2 & b^2 \end{bmatrix}\right) = ab\det\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & a-1 & b-1 \\ 0 & a^2-1 & b^2-1 \end{bmatrix}\right) = ab\det\left(\begin{bmatrix} a-1 & b-1 \\ a^2-1 & b^2-1 \end{bmatrix}\right)$$

$$= ab(a-1)(b-1)\det\left(\begin{bmatrix} 1 & 1 \\ a+1 & b+1 \end{bmatrix}\right) = ab(a-1)(b-1)(b-a).$$

Therefore, the matrix is nonsingular if and only if $a, b \neq \{0, 1\}$ and $a \neq b$.

Consider the vector space P_3. For a polynomial p, let p' denote its derivative.

a. Show that the set
$$\{p(x) \in P_3 \mid p(1) = p'(1)\}$$
is a subspace of P_3.

b. Let $L : P_3 \to P_3$ be given by
$$L\big(p(x)\big) = p'(1)x + p(1).$$
Is L a linear transformation? If so, find its matrix representation with respect to the basis $\{1, x, x^2\}$.

REQUIRED KNOWLEDGE: **Vector space, subspace, linear transformation, matrix representation**

SOLUTION:

3a: Let
$$S := \{p(x) \in P_3 \mid p(1) = p'(1)\}.$$
As it contains the zero polynomial, S is non-empty. Let
$$p(x) \in S$$
and α be a scalar. This means that
$$p(1) = p'(1) = 0.$$
Note that
$$(\alpha p)(1) = \alpha p(1) = \alpha p'(1) = (\alpha p)'(1).$$
Hence, $(\alpha p)(x) \in S$. So, the set S is closed under scalar multiplication. In order to show that it is closed under vector addition as well, let
$$p(x),\ q(x) \in S.$$
Note that
$$(p+q)(1) = p(1) + q(1) = p'(1) + q'(1) = (p+q)'(1).$$
Therefore, $(p+q)(x) \in S$. Consequently, S is a subspace.

3b: Note that
$$L\big(\alpha p(x)\big) = (\alpha p)'(1)x + (\alpha p)(1) = \alpha p'(1)x + \alpha p(1) = \alpha L\big(p(x)\big)$$
and
$$L\big(p(x) + q(x)\big) = (p+q)'(1)x + (p+q)(1) = p'(1)x + q'(1)x + p(1) + q(1) = L\big(p(x)\big) + L\big(q(x)\big).$$
Therefore, L is a linear transformation.

In order to find the matrix representation, we first apply L to the basis vector:
$$\begin{aligned} L(1) &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ L(x) &= x + 1 = 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \\ L(x^2) &= 2x + 1 = 1 \cdot 1 + 2 \cdot x + 0 \cdot x^2. \end{aligned}$$
This results in the following matrix representation:
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

4 (15 pts) **Least squares problem**

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ be with $\operatorname{rank}(A) = n$. Suppose that $\hat{x}$ is the solution for the least squares problem given by A and b. Show that

$$\|A\hat{x} - b\|^2 = b^T\big(I_m - A(A^TA)^{-1}A^T\big)b.$$

REQUIRED KNOWLEDGE: **Least squares problem, normal equations.**

SOLUTION:

When A is full column rank, the unique solution to the normal equations are given by

$$\hat{x} = (A^TA)^{-1}A^Tb.$$

Then, we have

$$A\hat{x} - b = A(A^TA)^{-1}A^Tb - b = \big(A(A^TA)^{-1}A^T - I_m\big)b.$$

This leads to

$$\begin{aligned}\|A\hat{x} - b\|^2 &= (A\hat{x} - b)^T(A\hat{x} - b) = \Big[\big(A(A^TA)^{-1}A^T - I_m\big)b\Big]^T\big(A(A^TA)^{-1}A^T - I_m\big)b \\ &= b^T\big(A(A^TA)^{-1}A^T - I_m\big)\big(A(A^TA)^{-1}A^T - I_m\big)b \\ &= b^T\big(I_m - A(A^TA)^{-1}A^T\big)b.\end{aligned}$$

5 $(8+7=15 \text{ pts})$ **Eigenvalues and diagonalization**

Let u, $v \in \mathbb{R}^n$ be nonzero vectors and $M = uv^T$.

a. Find all eigenvalues of M.

b. Show that M is diagonalizable if and only if $u^T v \neq 0$.

Required Knowledge: **Eigenvalues, diagonalization.**

Solution:

5a: Let λ and $x \neq 0$ be such that

$$uv^T x = \lambda x. \tag{1}$$

Therefore, we obtain

$$v^T uv^T x = \lambda v^T x \tag{2}$$

by multiplying both sides by v^T from left. Note that if $v^T x = 0$ then λ must be zero due to (1). If $v^T x \neq 0$ then (2) results in

$$\lambda v^T u.$$

Therefore, an eigenvalue of M is either zero or $v^T u$.

5b: Note that the null space of M is the same as the null space of v^T since $u \neq 0$. As $v \neq 0$, we have $\dim N(M) = n-1$ from the rank-nullity theorem. Therefore, M is diagonalizable if and only if zero eigenvalue has multiplicity $n-1$. This happens if and only if $v^T u \neq 0$.

Let $a \in \mathbb{R}$ and M be a 4×4 matrix with the characteristic polynomial $p_M(\lambda) = (\lambda^2 - a^2)(\lambda^2 + a^2)$.

a. Determine the eigenvalues of M.

b. Determine the determinant of M.

c. Find all values of a such that M is nonsingular.

d. Determine trace(M), the sum of the diagonal entries.

e. Show that M is diagonalizable if $a \neq 0$.

f. Let $a = 0$. Find such an M matrix that is diagonalizable. Also, find such an M matrix that is not diagonalizable.

Required Knowledge: **Eigenvalues, characteristic polynomial, nonsingularity, determinant, trace, diagonalizability.**

Solution:

6a: Eigenvalues are the roots of the characteristic polynomial. As such, we have

$$\lambda_{1,2} = \pm a \quad \text{and} \quad \lambda_{3,4} = \pm ia$$

.

6b: Determinant is the product of eigenvalues. Therefore, $\det(M) = p_M(0) = -a^4$.

6c: A square matrix A is nonsingular if and only if $\det(A) \neq 0$. Therefore, M is nonsingular if and only if $a \neq 0$.

6d: Trace of a square matrix is the sum of its eigenvalues. As such, trace(M) $= 0$.

6e: If $a \neq 0$, then M has distinct eigenvalues. As such, it is diagonalizable.

6f: Consider the matrices

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that $p_{M_1}(\lambda) = p_{M_2}(\lambda) = \lambda^4$. Clearly, M_1 is diagonalizable. We claim that M_2 is diagonalizable. This would follows from the fact that the null space of M_2 is of dimension 1 although its zero (only) eigenvalue has multiplicity 4.